Abstract Linear Algebra

Problem Set 1

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1 The Complex Field $\mathbb{C}$

Definition 1.1. ***The Complex Field***: *A complex number is an ordered pair* (a, b)*, where* $a, b \in \mathbb{R}$*, and in which they are written as* $a + bi$*, where* $i = \sqrt{-1}$*. The set of all complex numbers is denoted by* $\mathbb{C}$*:*

$$\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}.$$

Definition 1.2. ***Arithmetic on*** $\mathbb{C}$*:*

Addition *on* $\mathbb{C}$ *is defined by*

$$(a + bi) + (c + di) = (a + c) + (b + d)i. \tag{1}$$

Multiplication *on* $\mathbb{C}$ *is defined by*

$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i. \tag{2}$$

1. Show that $\alpha + \beta = \beta + \alpha$ for all $\alpha, \beta \in \mathbb{C}$.

Soln: Direct Proof: Let $\alpha, \beta \in \mathbb{C}$.

$$\alpha = a + bi \quad \text{and} \quad \beta = c + di,$$

where $a, b, c, d \in \mathbb{R}$. We recall that addition on $\mathbb{C}$ (1.2) is defined as:

$$\begin{aligned} \alpha + \beta &= (a + bi) + (c + di) \\ &= (a + c) + (b + d)i. \\ \beta + \alpha &= (c + di) + (a + bi), \\ &= (c + a) + (d + b)i. \end{aligned}$$

Since $a, b, c, d \in \mathbb{R}$, and addition is *commutative* in $\mathbb{R}$, we can write

$$(a + c) + (b + d)i = (c + a) + (d + b)i,$$

and thus, $\alpha + \beta = \beta + \alpha$.

□

2. Show that for every $\alpha \in \mathbb{C}$, there exists a unique $\beta \in \mathbb{C}$ such that $\alpha + \beta = 0$.

Soln: Proof By Contradiction: Assume there exists two distinct values $\beta_1, \beta_2 \in \mathbb{C}$ such that

$$\alpha + \beta_1 = 0 \quad \text{and} \quad \alpha + \beta_2 = 0.$$

We can subtract the equations from each other,

$$(\alpha + \beta_1) - (\alpha + \beta_2) = 0 - 0,$$

which when simplifying, yields

$$\beta_1 - \beta_2 = 0.$$

This leads to a contradiction, since it implies that $\beta_1 = \beta_2$. Therefore, the β value such that $\forall \alpha \in \mathbb{C} : \alpha + \beta = 0$ must be unique.

□

3. Show that for every $\alpha \in \mathbb{C}$ with $\alpha \neq 0$, there exists a unique $\beta \in \mathbb{C}$ such that $\alpha\beta = 1$.

Soln: Proof By Contradiction: Assume there exists two distinct values $\beta_1, \beta_2 \in \mathbb{C}$ such that

$$\alpha\beta_1 = 1 \quad \text{and} \quad \alpha\beta_2 = 1.$$

Then, subtract the two equations

$$\alpha\beta_1 - \alpha\beta_2 = 1 - 1$$
$$\alpha(\beta_1 - \beta_2) = 0.$$

Since we have $\alpha \neq 0$, we can divide both sides by α, yielding

$$\beta_1 - \beta_2 = 0.$$

This implies $\beta_1 = \beta_2$, which gives rise to a contradiction. Therefore, the β value such that $\forall \alpha \in \mathbb{C} : \alpha\beta = 1$ must be unique.

□

4. Show that
$$\frac{-1+\sqrt{3}i}{2}$$
is a cube root of 1 (meaning that its cube equals 1).

Soln: Direct Proof: Let $z = \frac{-1+\sqrt{3}i}{2}$. From 1.2, we know that multiplication on $\mathbb{C}$ is defined as
$$\alpha\beta = (a+bi)(c+di) = (ac-bd) + (ad+bc)i;$$
where $\alpha = a+bi, \beta = c+di$, and $a,b,c,d \in \mathbb{R}$. Note that, when multiplying z by itself, $a = c$ and $d = b$. For z, we have $a = c = -\frac{1}{2}$, and $b = d = \frac{\sqrt{3}}{2}$. We'll start by computing z^2, and then compute $(z^2)(z)$.

$$\begin{aligned}
z^2 &= (ac-bd) + (ad+bc)i \\
&= \left(\left(-\frac{1}{2}\right)^2 - \left(\frac{\sqrt{3}}{2}\right)^2\right) + \left(\left(-\frac{1}{2}\cdot\frac{\sqrt{3}}{2}\right) + \left(\frac{\sqrt{3}}{2}\cdot\frac{-1}{2}\right)\right)i \\
&= \left(\frac{1}{4} - \frac{3}{4}\right) + \left(-\frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{4}\right)i \\
&= -\frac{1}{2} - \frac{\sqrt{3}i}{2} \\
&= \frac{-1-\sqrt{3}i}{2}
\end{aligned}$$

Now we take z^2 from above and compute the product $z^3 = z^2 z$, with care to use the right a, b, c and d values:

$$\begin{aligned}
z^3 &= \left(\left(-\frac{1}{2}\right)^2 - \left(-\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2}\right)\right) + \left(\left(-\frac{1}{2}\cdot\frac{\sqrt{3}}{2}\right) + \left(-\frac{\sqrt{3}}{2}\cdot\frac{-1}{2}\right)\right) \\
&= \left(\frac{1}{4} + \frac{3}{4}\right) + \left(-\frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4}\right) \\
&= 1 + 0 \\
&= 1.
\end{aligned}$$

□

5. Find two distinct square roots of i.

> **Soln: TODO**:

6. Explain why there does not exist $\lambda \in \mathbb{C}$ such that

$$\lambda(2-3i, 5+4i, -6+7i) = (12-5i, 7+22i, -32-9i).$$

Soln: Proof by Contradiction: Assume $\exists \lambda \in \mathbb{C}$ such that

$$\lambda \begin{pmatrix} 2-3i \\ 5+4i \\ -6+7i \end{pmatrix} = \begin{pmatrix} 12-5i \\ 7+22i \\ -32-9i \end{pmatrix}.$$

If such a λ exists, then it must satisfy the following system of complex equations:

$$\begin{cases} \lambda(2-3i) & = 12-5i \\ \lambda(5+4i) & = 7+22i \\ \lambda(-6+7i) & = -32-9i \end{cases}$$

We can solve for λ from the first equation,

$$\begin{aligned} \lambda(2-3i) &= 12-5i \\ \lambda &= \frac{12-5i}{2-3i} \\ \lambda &= 3+2i, \end{aligned}$$

and check if it satisfies the second equation:

$$\begin{aligned} (3+2i)(5+4i) &= 15+12i+10i-8 \\ &= 7+22i. \end{aligned}$$

Thus, this λ satisfies the first two equations. Lastly, we check the third:

$$\begin{aligned} (3+2i)(-6+7i) &= -18+21i-12i-14 \\ &= -32+9i \neq -32-9i \end{aligned}$$

and arrive at a contradiction. We conclude that no single $\lambda \in \mathbb{C}$ can satisfy all three equations simultaneously.

□

2 Vector Spaces

Definition 2.1. *A vector space V over a field F is a set along with an addition and a scalar multiplication*

$$+ : V \times V \to V \quad \text{and} \quad \cdot : F \times V \to V$$

such that the following properties hold:

- ***Commutativity**: $u + v = v + u$ for all $u, v \in V$.*
- ***Associativity**: $(u + v) + w = u + (v + w)$ and $a(bv) = (ab)v$ for all $u, v, w \in V$ and all $a, b \in F$.*
- ***Additive Inverse**: For every $v \in V$, there exists $-v \in V$ such that $v + (-v) = 0$.*
- ***Additive Identity**: There exists $0 \in V$ such that $v + 0 = v$ for all $v \in V$.*
- ***Multiplicative Identity**: There exists $1 \in V$ such that $1v = v$ for all $v \in V$.*
- ***Distributive Properties**: $a(u + v) = au + av$ and $(a + b)v = av + bv$ for all $a, b \in F$ and all $u, v \in V$.*

Theorem 2.2. *Any subspace of a vector space is a vector space.*

1. Prove that $-(-v) = v$ for every $v \in V$ (where V is a vector space).

Soln: By definition of a vector space (2.1), the additive inverse $-v$ of $v \in V$ satisfies:

$$v + (-v) = 0. \tag{3}$$

Considering the element $-(-v)$, which is the additive inverse of $-v$, and by the properties of the additive inverse, we have:

$$(-v) + (-(-v)) = 0. \tag{4}$$

From (Eq. 3), we know that $v + (-v) = 0$. Let's add v to both sides of (Eq. 4):

$$v + ((-v) + (-(-v))) = v + 0.$$

By the property of the additive identity $v + 0 = v$. Since addition on V is associative, we can rewrite the left-hand side:

$$(v + (-v)) + (-(-v)) = v.$$

From $v + (-v) = 0$, we get:

$$\begin{aligned} 0 + (-(-v)) &= v \\ -(-v) &= v. \end{aligned}$$

Thus, for every $v \in V$, we have $-(-v) = v$.

□

2. Suppose $a \in F, v \in V$, and $av = 0$. Prove that $a = 0$ or $v = 0$.

Soln: If $a = 0$, the equation $av = 0$ is trivially true for any $v \in V$. Assume $a \neq 0$ and that $av = 0$ for some $v \in V$. Since $a \in F$ and $a \neq 0$, the field F guarantees the existing of an inverse a^{-1} such that $a^{-1}a = 1$. Using this, we multiply both sides of the equation $av = 0$ by a^{-1}:

$$a^{-1}(av) = a^{-1} \cdot 0.$$

By the associativity of scalar multiplication on V, we have:

$$(a^{-1}a)v = 0,$$

and since $a^{-1}a = 1$, this simplifies to:

$$1 \cdot v = 0.$$

The scalar 1 is the multiplicative identity in the field F, so $1 \cdot v = v$. Therefore, the equation becomes:

$$v = 0.$$

Thus, if $a \neq 0$, we have shown that $v = 0$. Therefore, either $a = 0$ or $v = 0$.

□

3. Suppose $v, w \in V$. Explain why there exists a unique $x \in V$ such that $v + 3x = w$.

Soln: Existence: To prove existence of x, we can use the axioms of vector spaces (2.1), particularly, the properties of scalar multiplication and additive inverses. Subtract v from both sides of the equation:

$$\begin{aligned}(v + 3x) - v &= w - v \\ 3x &= w - v. \end{aligned} \tag{5}$$

Since $3 \in F$ is a non-zero scalar and F, the field over which the vector space is defined, has multiplicative inverses for non-zero elements, there exists $\frac{1}{3} \in F$:

$$x = \frac{1}{3}(w - v).$$

Thus, we have proven the existence of an $x \in V$ that satisfies $v + 3x = w$.

Uniqueness: Suppose there are two values $x_1, x_2 \in V$ such that they satisfy

$$v + 3x_1 = w \quad \text{and} \quad v + 3x_2 = w.$$

Combining the two equations, we have:

$$\begin{aligned}(v + 3x_1) - (v + 3x_2) &= w - w \\ 3x_1 - 3x_2 &= 0. \end{aligned} \tag{6}$$

We can factor out the scalar 3,

$$3(x_1 - x_2) = 0,$$

which implies $x_1 - x_2 = 0$, and in turn, $x_1 = x_2$. Therefore x is unique.

□

4. The empty set is not a vector space. The empty set fails to satisfy only one of the requirements listed in the definition of a vector space (1.20). Which one?

Soln: The **additive identity** axiom in (1.20) begins "*there exists an element* $0 \in V$...". However, the empty set $\emptyset$ **does not contain any elements**. Therefore it fails to satisfy this axiom.

□

5. Show that in the definition of a vector space (1.20), the additive inverse condition can be replaced with the condition that

$$\forall v \in V : 0v = 0.$$

The phrase a "condition can be replaced" in a definition means that the collection of objects satisfying the definition is unchanged if the original condition is replaced with the new condition.

Soln: **Direct Proof**: Let $v \in V$. Then

$$\begin{aligned} 0 &= 0v \\ &= (1 + (-1))v \\ &= 1v + (-1)v \\ &= v + (-1)v. \end{aligned}$$

This $(-1)v$ is an additive inverse of v. Hence the additive inverse condition is indirectly satisfied by the condition we replaced it with.

□

6. Suppose S is a nonempty set. Let V^S denote the set of functions from S to V. Define a natural addition and scalar multiplication on V^S, and show that V^S is a vector space with these definitions.

Soln: For $f, g \in V^S$, $x \in S$, and $\lambda \in F$, let $f + g$ and λf be the operations defined by

$$(f + g)(x) = f(x) + g(x) \quad \text{and} \quad (\lambda f)(x) = \lambda f(x).$$

the additive identity of V^S is the function from S to V that is identically 0, and the additive inverse of $f \in V^S$ is the function from S to V such that $x \mapsto -f(x)$.

□

3 Subspaces

Definition 3.1. ***Conditions of a Subspace*** *A subset $U \subset V$ is a subspace of V if, and only if, U satisfies the following properties.*

- ***additive identity***
$$0 \in U.$$
- ***closed under addition***
$$u, w \in U \implies u + w \in W.$$
- ***closed under scalar multiplication***
$$a \in F \text{ and } u \in U \implies au \in U.$$

1. For each of the following subsets of F^3, determine whether or not they are a subspace of F^3.
   (a) $W = \{(x_1, x_2, x_3) \in F^3 : x_1 + 2x_2 + 3x_3 = 0\}$
   (b) $W = \{(x_1, x_2, x_3) \in F^3 : x_1 + 2x_2 + 3x_3 = 4\}$
   (c) $W = \{(x_1, x_2, x_3) \in F^3 : x_1 x_2 x_3 = 0\}$
   (d) $W = \{(x_1, x_2, x_3) \in F^3 : x_1 = 5x_3\}$

Soln: We will check each W to see if they satisfy the criteria for a subspace as defined in 3.1.

(a) $W = \{(x_1, x_2, x_3) \in F^3 : x_1 + 2x_2 + 3x_3 = 0\}$.

Additive Identity: $W = \{(x_1, x_2, x_3) \in F^3 : x_1 + 2x_2 + 3x_3 = 0\}$ contains the *additive identity*, since $(0, 0, 0)$ satisfies $x_1 + 2x_2 + 3x_3 = 0$.

Closed Under Addition: Suppose $u = (x_1, x_2, x_3) \in W$ and $v = (y_1, y_2, y_3) \in W$, where each satisfy $x_1 + 2x_2 + 3x_3 = 0$ and $y_1 + 2y_2 + 3y_3 = 0$. Then:

$$(x_1 + y_1) + 2(x_2 + y_2) + 3(x_3 + y_3) = (x_1 + 2x_2 + 3x_3) + (y_1 + 2y_2 + 3y_3) = 0.$$

Hence, the subset is closed under addition.

Closed Under Scalar Mult.: Let $u = (x_1, x_2, x_3) \in W$ and $c \in F$. Then:

$$c(x_1 + 2x_2 + 3x_3) = c(0) = 0.$$

Therefore, $cu \in W$. We conclude that this W **is a subspace** of F^3.

□

(b) $W = \{(x_1, x_2, x_3) \in F^3 : x_1 + 2x_2 + 3x_3 = 4\}$.

Additive Identity: The subset $W = \{(x_1, x_2, x_3) \in F^3 : x_1 + 2x_2 + 3x_3 = 4\}$ **does not** contain the additive identity, since the zero vector does not satisfy $x_1 + 2x_2 + 3x_3 = 4$:

$$0 + 2(0) + 3(0) \neq 4.$$

Therefore, the subset **is not a subspace** of F^3.

□

(c) **TODO**

(d) **TODO**

2. Show that the set of differentiable real-valued functions f on the interval $(-4, 4)$ such that $f'(-1) = 3f(2)$ is a subspace of $\mathbb{R}^{(-4,4)}$.

Soln: Let S be the set of differentiable real-valued functions on the interval $(-4, 4)$ such that $f'(-1) = 3f(2)$. We will check the subspace criteria (3.1) one-by-one.

Additive Identity: For the zero function, $f(x) = 0$, we have $f'(x) = 0$ for all x, including f'(-1)=0, and $f(2) = 0$. Thus, $f'(-1) = 0 = 3f(2)$, so the zero function satisfies the condition $f'(-1) = 3f(2)$, and therefore, S contains the zero function; $f(x) = 0 \in S$.

Closed Under Addition: Let $f, g \in S$, e.g., $f'(-1) = 3f(2)$ and $g'(-1) = 3g(2)$. We need to show that $(f + g) \in S$, meaning $(f + g)'(-1) = 3(f + g)(2)$.

By the linearity of differentiation, we have:

$$(f + g)'(-1) = f'(-1) + g'(-1)$$

and

$$(f + g)(2) = f(2) + g(2).$$

Using the fact that $f'(-1) = 3f(2)$ and $g'(-1) = 3g(2)$, we get:

$$(f + g)'(-1) = f'(-1) + g'(-1) = 3f(2) + 3g(2) = 3(f(2) + g(2)) = 3(f + g)(2).$$

Thus, $(f + g)'(-1) = 3(f + g)(2)$, which shows that $(f + g) \in S$.

Closed Under Scalar Mult.: Let $f \in S$ and $c \in R$. We want to show that $(cf) \in S$, meaning $(cf)'(-1) = 3(cf)(2)$. By the linearity of differentiation and scalar multiplication, we have:

$$(cf)'-1 = cf'(-1)$$

and

$$(cf)(2)cf(2).$$

Using the face that $f'(-1) = 3f(2)$, we get:

$$(cf)'(-1) = cf'(-1) = c \cdot 3f(2) = 3(cf(2)).$$

Thus, $(cf)'(-1) = 3(cf)(2)$, which shows that $cf \in S$.

The set S of differentiable real-valued functions f on the interval $(-4, 4)$, with $f'(-1) = 3f(2)$, satisfies all the criteria of a subspace of $\mathbb{R}^{(-4,4)}$.

□

3. (a) Is $\{(a, b, c) \in \mathbb{R}^3 : a^3 = b^3\}$ a subspace of $\mathbb{R}^3$?
   (b) Is $\{(a, b, c) \in \mathbb{C}^3 : a^3 = b^3\}$ a subspace of $\mathbb{C}^3$?

Soln:

(a) Let $S = \{(a, b, c) \in \mathbb{R}^3 : a^3 = b^3\}$. For S to be a subspace of $\mathbb{R}^3$ it must satisfy the properties listed in 3.1.

Additive Identity: The additive identity (zero function) $(0, 0, 0)$ exists in S, because it satisfies the equation

$$a^3 = b^3 \implies 0^3 = 0^3.$$

Closed Under Addition: Let $\alpha = (a_1, b_1, c_1) \in S$ and $\beta = (a_2, b_2, c_2) \in S$. These vectors satisfy the condition that:

$$a_1^3 = b_1^3 \quad \text{and} \quad a_2^3 = b_2^3. \tag{7}$$

The definition of addition on S gives us

$$\alpha + \beta = \begin{pmatrix} a_1 + a_2 \\ b_1 + b_2 \\ c_1 + c_2 \end{pmatrix}.$$

The condition for $\alpha + \beta \in S$ is:

$$(a_1 + a_2)^3 = (b_1 + b_2)^3. \tag{8}$$

Now, let's check whether the equation $(a_1 + a_2)^3 = (b_1 + b_2)^3$ holds in general. Expanding both sides using the binomial theorem:

$$(a_1 + a_2)^3 = a_1^3 + 3a_1^2 a_2 + 3a_1 a_2^2 + a_2^3,$$

and

$$(b_1 + b_2)^3 = b_1^3 + 3b_1^2 b_2 + 3b_1 b_2^2 + b_2^3.$$

For the sum $\alpha + \beta$ to satisfy (8), we need:

$$a_1^3 + 3a_1^2 a_2 + 3a_1 a_2^2 + a_2^3 = b_1^3 + 3b_1^2 b_2 + 3b_1 b_2^2 + b_2^3.$$

However, we only know that $a_1^3 = b_1^3$ and $a_2^3 = b_2^3$; we know nothing about the remaining terms. Thus in general, and ignoring special cases where the other terms are 0,

$$(a_1 + a_2)^3 \neq (b_1 + b_2)^3.$$

Since $(a_1 + a_2)^3 \neq (b_1 + b_2)^3$ in general, the sum of two vectors in S does not necessarily belong to S. Thus, S is **not closed under addition**, and S is **not a subspace** of $\mathbb{R}^3$.

□

4. Prove or give a counterexample: If U is a nonempty subset of $\mathbb{R}^2$ such that U is closed under addition and under taking additive inverses (meaning $-u \in U$ whenever $u \in U$), then U is a subspace of $\mathbb{R}^2$.

Soln: Let U be a nonempty subset of $\mathbb{R}^2$ such that it satisfies the properties listed above. The missing property we need to show for U is being closed under scalar multiplication. Consider $c = -1 \in \mathbb{R}$. If $u \in U$, then $-u \in U$ by the assumption that U is closed under taking additive inverses. So scalar multiplication by -1 is already implied by this property.

However, consider the example:

$$U \subset \mathbb{R}^2 = \{(x, y) \in \mathbb{R}^2 : x, y \in \mathbb{Z}\}.$$

This U is **not closed under scalar multiplication**. For example, if $u = (1, 0) \in U$ and we multiply by a scalar $r = \frac{1}{2} \in \mathbb{R}$, we get:

$$\frac{1}{2}(1, 0) = \left(\frac{1}{2}, 0\right),$$

which is not in U, since $\frac{1}{2} \notin \mathbb{Z}$.

Thus, U is **not a subspace** of $\mathbb{R}^2$, even though it is closed under addition and additive inverses, because it fails to satisfy the property of being closed under scalar multiplication.

□

5. Give an example of a nonempty subset U of $\mathbb{R}^2$ such that U is closed under scalar multiplication, but U is not a subspace of $\mathbb{R}^2$.

6. A function $f : \mathbb{R} \to \mathbb{R}$ is called *periodic* if there exists a positive number p such that $f(x) = f(x + p)$ for all $x \in \mathbb{R}$. Is the set of periodic functions from $\mathbb{R}$ to $\mathbb{R}$ a subspace of $\mathbb{R}^{\mathbb{R}}$? Explain.

7. Suppose V_1 and V_2 are subspaces of V. Prove that the intersection $V_1 \cap V_2$ is a subspace of V.

8. Prove that the intersection of every collection of subspaces of V is a subspace of V.

9. Prove that the union of two subspaces of V is a subspace of V if, and only if, one of the subspaces is contained in the other

10. Suppose

$$U = \{(x, -x, 2x) \in F^3 : x \in F\} \quad \text{and} \quad W = \{(x, x, 2x) \in F^3 : x \in F\}.$$

Describe $U + W$ using symbols, and give a description of $U + W$ that uses no symbols.

11. Suppose U is a subspace of V. What is $U + U$?

12. Prove or give a counterexample: If V_1, V_2, U are subspaces of V such that

$$V_1 + U = V_2 + U, \tag{9}$$

then $V_1 = V_2$.

13. Suppose

$$U = \{(x, x, y, y) \in F^4 : x, y \in F\}. \tag{10}$$

Find a subspace W of F^4 such that $F^4 = U \oplus W$. Where $\oplus$ denotes a *direct sum*.

14. A function $f : \mathbb{R} \to \mathbb{R}$ is called *even* if

$$f(-x) = f(x) \tag{11}$$

for all $x \in \mathbb{R}$. A function $f : \mathbb{R} \to \mathbb{R}$ is called *odd* if

$$f(-x) = -f(x) \tag{12}$$

for all $x \in \mathbb{R}$. Let V_e denote the set of real-valued even functions on $\mathbb{R}$ and let V_o denote the set of real-valued odd functions on $\mathbb{R}$. Show that $\mathbb{R}^{\mathbb{R}} = V_e \oplus V_o$.